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# Existence,multiplicity, and nonexistence of solutions for a $p$-Kirchhoff elliptic equation on $\mathbb{R}^N$

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**Abstract**

In this paper, we study the multiplicity of solutions for the following nonhomogeneous $p$-Kirchhoff elliptic equation:

$$\left(a+\lambda\left(\int_{\mathbb{R}^N}(|\nabla u|^p+|u|^p)\,dx\right)^m\right)(-\Delta_p u+|u|^{p-2}u)=f(u)+h(x),\quad x\in\mathbb{R}^N,\tag{0.1}$$

with $a,\lambda,m>0$ and $1<p<N$. By variational methods we prove that problem (0.1) admits at least two solutions under appropriate assumptions on $f(u)$ and $h(x)$. The main difficulty to overcome is the lack of an a priori bound for Palais-Smale sequence. Motivated by Jeanjean (Proc. R. Soc. Edinb., Sect. A 129:787-809, 1999), we use a cut-off functional to obtain a bounded $(PS)$ sequence. Also, if $f(u)=|u|^{q-2}u$, $p<q<\min\{p(m+1),p^*=\frac{pN}{N-p}\}$, and $h(x)=0$, then we prove that problem (0.1) has at least one nontrivial solution for any $\lambda\in(0,\lambda^*]$ and has no nontrivial weak solutions for any $\lambda\in(\lambda^*,+\infty)$.

**Keywords:** $p$-Kirchhoff elliptic equation; bounded potential; variational methods; mountain pass lemma

## 1 Introduction

In this paper, we are interested in the multiplicity of solutions to the following nonhomogeneous $p$-Kirchhoff elliptic problem:

$$\left(a+\lambda\left(\int_{\mathbb{R}^N}(|\nabla u|^p+|u|^p)\,dx\right)^m\right)(-\Delta_p u+|u|^{p-2}u)=f(u)+h(x),\quad x\in\mathbb{R}^N,\tag{1.1}$$

where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator, and the nontrivial function $h(x)$ can be seen as a perturbation term. Problem (1.1) is a generalization of the model introduced by Kirchhoff [2]. More precisely, Kirchhoff proposed the model given by the equation

$$\rho_{tt}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L u_x^2\,dx\right)u_{xx}=0,\quad 0<x<L,t>0,\tag{1.2}$$

which takes into account the changes in length of string produced by transverse vibration. The parameters in (1.2) have the following meaning: $L$ is the length of the string, $h$ is the area of cross-section, $E$ is the Young modulus of material, $\rho$ is the mass density, and $P_0$ is the initial tension.

The equation

$$\rho_{tt} - M\big(\|\nabla u\|_2^2\big)\Delta u = f(x,u), \quad x \in \Omega, t > 0, \tag{1.3}$$

generalizes equation (1.2), where $M : \mathbb{R}^+ \to \mathbb{R}$ is a given function, $\Omega$ is a domain of $\mathbb{R}^N$. The stationary counterpart of (1.3) is the Kirchhoff-type elliptic equation

$$-M\big(\|\nabla u\|_2^2\big)\Delta u = f(x,u), \quad x \in \Omega, t > 0. \tag{1.4}$$

Some classical and interesting results on Kirchhoff-type elliptic equations can be found, for example, in [3–9].

Particularly, Li et al. [10] considered the Kirchhoff-type problem

$$\left(a + \lambda\left(\int_{\mathbb{R}^N} \big(|\nabla u|^2 + b|u|^2\big)\,dx\right)\right)(-\Delta u + bu) = f(u), \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $N \geq 3$, with constants $a, b > 0$ and $\lambda \geq 0$ under the following assumptions:

$(H_1)$ $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $|f(t)| \leq C(1 + t^{q-1})$ for all $t \in \mathbb{R}^+ = [0, +\infty)$ and some $q \in (2, 2^*)$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$;

$(H_2)$ $\lim_{t\to 0} \frac{f(t)}{t} = 0$; $\lim_{t\to +\infty} \frac{f(t)}{t} = +\infty$.

It is easy to see that $f(u) = |u|^{q-2}u$, $2 < q < 4$, and $N = 3$ satisfy these conditions. They obtained that there exists $\lambda_0 > 0$ such that, for any $\lambda \in [0, \lambda_0)$, problem (1.5) has at least one positive solution in $W^{1,2}(\mathbb{R}^N)$. The $\lambda_0$ depends on $f$, $a$, $b$, the Sobolev constant, and several test functions in [10]; it is not very clear whether the the existence of solutions for (1.5) still holds for large $\lambda > 0$. Recently, Chen et al. [11] studied the existence of positive solutions to the $p$-Kirchhoff problem

$$\begin{cases} (a + \lambda(\int_{\mathbb{R}^N} (|\nabla u|^p + b|u|^p)\,dx)^\tau)(-\Delta_p u + b|u|^{p-2}u) \\ \quad = |u|^{m-2}u + \mu|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ u(x) > 0, \quad x \in \mathbb{R}^N, \qquad u(x) \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.6}$$

where $a, b > 0$, $\tau, \lambda \geq 0$, $\mu \in \mathbb{R}$, and $1 < p < N$. By the Nehari manifold method, they proved that problem (1.6) admits at least a positive ground state solution for any $\lambda > 0$ when $p(\tau + 1) < q < m < p^* = \frac{pN}{N-p}$. However, does the existence of solutions for (1.5) still hold for any $\lambda > 0$ when $p < q < p(\tau + 1)$ and $\mu = 0$? This is a interesting problem. In this paper, we answer positively this question. More interesting results for Kirchhoff-type problems can be found in [1, 2, 5–7, 10–14].

In the present paper, we are ready to extend the analysis to the nonhomogeneous $p$-Kirchhoff-type equation of (1.1) in $\mathbb{R}^N$ with the nonlinearity $f(u)$ satisfying the following conditions:

$(F_1)$ $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $|f(t)| \le C(t^{p-1} + t^{q-1})$ for all $t \in \mathbb{R}^+$ and some $q \in (p, p^*)$, where $p^* = pN/(N-p)$, $1 < p < N$;
$(F_2)$ $\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0$;
$(F_3)$ $\lim_{t\to +\infty} \frac{f(t)}{t^{p-1}} = +\infty$.

In addition, we suppose that the nontrivial and nonnegative function $h(x) \equiv h(|x|) \in C^1(\mathbb{R}^N) \cap L^{p'}(\mathbb{R}^N)$ satisfies

$(H)$ there exists $\xi(x) \in L^{p'}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that

$$\left|\nabla h(x) \cdot x\right| \le \xi^{p'}(x), \quad \forall x \in \mathbb{R}^N, \tag{1.7}$$

with $p' = \frac{p}{p-1}$.

We will use the Ekeland variational principle [15] and a version of the mountain pass theorem in [1] to study the existence of multiple solutions of problem (1.1) in $\mathbb{R}^N$. It is well known that an important technical condition to get a bounded $(PS)$ sequence is the following Ambrosetti-Rabinowitz-type condition (AR): there exists $\theta > p$ such that $0 < \theta F(s) \le sf(s)$ for $s > 0$. The loss of (AR) condition renders variational techniques more delicate. Inspired by [1, 10], we use a cut-off functional and obtain a bounded $(PS)$ sequence.

In order to state our main result, we introduce some Sobolev spaces and norms. Let $W^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p + |u|^p \, dx\right)^{\frac{1}{p}}, \quad 1 < p < \infty. \tag{1.8}$$

We denote by $\|\cdot\|_q$ the usual $L^q(\mathbb{R}^N)$ norm. Then it well known that the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for $q \in (p, p^*]$ and there exists a constant $S_q$ such that

$$\|u\|_q \le S_q \|u\|, \quad \forall u \in W^{1,p}\big(\mathbb{R}^N\big). \tag{1.9}$$

Let $X = W_r^{1,p}(\mathbb{R}^N)$ be the subspace of $W^{1,p}(\mathbb{R}^N)$ containing only the radial functional. Then by the Lemma 2.2 in [11] we have that the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $q \in (p, p^*)$.

A function $u \in X$ is said to be a weak solution of (1.1) if for all $v \in X$,

$$\big(a + \lambda\|u\|^{pm}\big) \int_{\mathbb{R}^N} \big(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\big)\, dx = \int_{\mathbb{R}^N} \big(f(u) + h\big) v \, dx. \tag{1.10}$$

Let $I(u) : X \to \mathbb{R}$ be the energy functional associated with problem (1.1) defined by

$$I(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\|u\|^{p(m+1)} - \int_{\mathbb{R}^N} \big(F(u) + hu\big)\, dx, \tag{1.11}$$

where $F(u) = \int_0^u f(s)\, ds$. It is easy to see that the functional $I \in C^1(X, \mathbb{R})$ and its Gateaux derivative is given by

$$\begin{aligned} I'(u)v = {} & \big(a + \lambda\|u\|^{pm}\big) \int_{\mathbb{R}^N} \big(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\big)\, dx \\ & - \int_{\mathbb{R}^N} \big(f(u) + h\big) v\, dx, \quad \forall v \in X. \end{aligned} \tag{1.12}$$

Clearly, we see that a weak solution of (1.1) corresponds to a critical point of the functional.

The main result in this paper is as follows.

**Theorem 1.1** *Let* $(F_1)$*-*$(F_3)$ *and* $(H)$ *hold. Then, there exist* $\lambda_0, \widetilde{m}_0 > 0$ *such that, for any* $\lambda \in [0, \lambda_0)$, (1.1) *has at least two nontrivial solutions in* $X$ *when* $\|h\|_{p'} < \widetilde{m}_0$.

Furthermore, consider $h(x) = 0$ and $f(x,u) = |u|^{q-2}u$, $p < q < \min\{p(m+1), p^*\}$, that is,

$$\left(a + \lambda\left(\int_{\mathbb{R}^N} \left(|\nabla u|^p + |u|^p\right) dx\right)^m\right)\left(-\Delta_p u + |u|^{p-2}u\right) = |u|^{q-2}u, \quad x \in \mathbb{R}^N. \tag{1.13}$$

We can now state the second main result.

**Theorem 1.2** *Let* $a > 0$ *and* $p < q < \min\{p(m+1), p^*\}$. *Then there exists* $\lambda^* > 0$ *such that problem* (1.13) *has at least one nontrivial solution for any* $\lambda \in (0, \lambda^*]$ *and has no nontrivial weak solutions for any* $\lambda \in (\lambda^*, +\infty)$.

**Remark 1.3** In [11], Chen and Zhu considered the case $p < p(m+1) < q < p^*$. They proved that problem (1.1) admits at least one positive solution for any $\lambda > 0$.

## 2 Proof of Theorem 1.1

In this section, we first establish some properties of the functional $I$ and then prove Theorem 1.1. Throughout the paper, we denote by $C$ or $C_i$ s positive constants that may vary from line to line and are not essential to the problem.

**Lemma 2.1** *If assumptions* $(F_1)$*-*$(F_3)$ *hold and* $h(x) \in L^{p'}(\mathbb{R}^N)$, *then there exist* $\rho, \alpha, m_0 > 0$ *such that* $I(u) \geq \alpha > 0$ *with* $\|u\| = \rho$ *and* $\|h\|_{p'} < m_0$.

*Proof* It follows from $(F_1)$-$(F_2)$ that

$$F(s) \leq \varepsilon|s|^p + C_\varepsilon|s|^q, \quad \forall s \in \mathbb{R}, \tag{2.1}$$

with $\varepsilon > 0$. By the Hölder inequality we have

$$\left|\int_{\mathbb{R}^N} hu\,dx\right| \leq S_q^{-1}\|h\|_{p'}\|u\| \leq \epsilon\|u\|^p + C_\epsilon\|h\|_{p'}^{p'}. \tag{2.2}$$

Thus,

$$\begin{aligned} I(u) &\geq \frac{a}{p}\|u\|^p - \varepsilon\|u\|^p - C_\varepsilon\|u\|^q - \epsilon\|u\|^p - C_\epsilon\|h\|_{p'}^{p'} \\ &\geq \frac{a}{2p}\|u\|^p - C_1\|u\|^q - C_2\|h\|_{p'}^{p'}, \end{aligned} \tag{2.3}$$

where $\varepsilon = \epsilon = \frac{a}{4p}$, $C_1$, $C_2$ are some positive constants. Let

$$z(t) = \frac{a}{2p}t^p - C_1 t^q, \quad t \geq 0. \tag{2.4}$$

We see that there exists $\rho > 0$ such that $\max_{t\geq 0} z(t) = z(\rho) \equiv m_0 > 0$. Then it follows from (2.3) that there exists $\alpha > 0$ such that $I(u) \geq \alpha$ with $\|u\| = \rho$ and $\|h\|_{p'} < m_0$. This ends the proof of Lemma 2.1. □

We denote by $B_r$ the open ball in $X$ centered at the origin with radius $r$. By Ekland's variational principle [15] we get the following lemma, which implies that there exists a function $u_0$ such that $I'(u_0) = 0$ and $I(u_0) < 0$ if $\|h\|_{p'}$ is small.

**Lemma 2.2** *Let assumptions* $(F_1)$-$(F_3)$ *hold, and* $h(x) \in L^{p'}(\mathbb{R}^N)$, $h(x) \not\equiv 0$, *with* $\|h\|_{p'} < m_0$. *Then there exists a function* $u_0 \in X$ *such that*

$$I(u_0) = \inf\bigl\{I(u) : u \in \overline{B}_\rho\bigr\} < 0, \tag{2.5}$$

*and* $u_0$ *is a nontrivial weak solution of problem* (1.1).

*Proof* Choose a function $\phi \in C_0^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} h(x)\phi(x)\,dx > 0$. Then

$$I(t\phi) \le \frac{a}{p}t^p\|\phi\|^p + \frac{\lambda}{p(m+1)}t^{p(m+1)}\|\phi\|^{p(m+1)} - t\int_{\mathbb{R}^N} h(x)\phi\,dx < 0 \tag{2.6}$$

for small $t > 0$ and thus for any open ball $B_\kappa \subset X$ such that $-\infty < c_\kappa = \inf_{\overline{B}_\kappa} I(u) < 0$. Thus,

$$c_\rho = \inf_{u\in\overline{B}_\rho} I(u) < 0 \quad \text{and} \quad \inf_{u\in\partial B_\rho} I(u) > 0, \tag{2.7}$$

where $\rho$ is given in Lemma 2.1. Let $\varepsilon_n \downarrow 0$ be such that

$$0 < \varepsilon_n < \inf_{u\in\partial B_\rho} I(u) - \inf_{u\in B_\rho} I(u). \tag{2.8}$$

Then, by Ekland's variational principle [15] there exists $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_\rho \le I(u_n) < c_\rho + \varepsilon_n \tag{2.9}$$

and

$$I(u_n) < I(u) + \varepsilon_n\|u_n - u\| \qquad \text{for all } u \in \overline{B_\rho}, u_n \ne u. \tag{2.10}$$

Then, it follows from (2.8)-(2.10) that

$$I(u_n) < c_\rho + \varepsilon_n \le \inf_{u\in B_\rho} I(u) + \varepsilon_n < \inf_{u\in\partial B_\rho} I(u). \tag{2.11}$$

So $u_n \in B_\rho$, and we now consider the function $F : \overline{B}_\rho \to \mathbb{R}$ given by

$$F(u) = I(u) + \varepsilon_n\|u_n - u\|, \quad u \in \overline{B}_\rho. \tag{2.12}$$

Then (2.10) shows that $F(u_n) < F(u)$, $u \in \overline{B}_\rho$, $u_n \ne u$, and thus $u_n$ is a strict local minimum of $F$. Moreover,

$$t^{-1}\bigl(F(u_n + tv) - F(u_n)\bigr) \ge 0 \quad \text{for small } t > 0, \forall v \in B_1. \tag{2.13}$$

Hence,

$$t^{-1}\big(I(u_n+tv)-I(u_n)\big)+\varepsilon_n\|v\|\geq 0. \tag{2.14}$$

Passing to the limit as $t\to 0^+$, it follows that

$$I'(u_n)v+\varepsilon_n\|v\|\geq 0, \quad \forall v\in B_1. \tag{2.15}$$

Replacing $v$ in (2.15) by $-v$, we get

$$-I'(u_n)v+\varepsilon_n\|v\|\geq 0, \quad \forall v\in B_1, \tag{2.16}$$

so that $\|I'(u_n)\|\leq\varepsilon_n$. Therefore, there is a sequence $\{u_n\}\in B\rho$ such that $I(u_n)\to c_\rho<0$ and $I'(u_n)\to 0$ in $X^*$ as $n\to\infty$. In the following, we will prove that $\{u_n\}$ has a convergent subsequence in $X$. Indeed, since $\|u_n\|<\rho$, by the reflexivity of $X$ and compact embedding $X\hookrightarrow L^q$ for all $q\in(p,p^*)$, passing to a subsequence, we can assume that

$$u_n\rightharpoonup u_0, \quad \text{in } X; \qquad u_n\to u_0, \quad L^q(\mathbb{R}^N); \qquad u_n\to u_0, \quad \text{a.e. in } \mathbb{R}^N. \tag{2.17}$$

By (1.12) we can get

$$\big(I(u_n)-I(u_0)\big)'(u_n-u_0)=P_n+Q_n+K_n, \tag{2.18}$$

where

$$\begin{aligned}
P_n &= \big(a+\lambda\|u_n\|^{pm}\big)\int_{\mathbb{R}^N}\big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\big)\nabla(u_n-u_0) \\
&\quad+\big(|u_n|^{p-2}u_n-u_0^{p-2}u_0\big)(u_n-u_0)\,dx, \\
Q_n &= \lambda\big(\big(\|u_n\|^{pm}-\|u_0\|^{pm}\big)\big)\int_{\mathbb{R}^N}|\nabla u_0|^{p-2}\nabla u_0\nabla(u_n-u_0) \\
&\quad+|u_0|^{p-2}u_0(u_n-u_0)\,dx, \\
K_n &= \int_{\mathbb{R}^N}\big(f(u_n)-f(u_0)\big)(u_n-u_0)\,dx.
\end{aligned} \tag{2.19}$$

It is clear that

$$\big(I(u_n)-I(u_0)\big)'(u_n-u_0)\to 0 \quad \text{as } n\to\infty. \tag{2.20}$$

By $(F_1)$ and $(F_2)$, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$\big|f(t)\big|\leq\varepsilon|t|^{p-1}+C_\varepsilon|t|^{q-1}, \quad t\in\mathbb{R}. \tag{2.21}$$

Hence,

$$\begin{aligned}
|K_n| &= \left|\int_{\mathbb{R}^N}\big(f(u_n)-f(u_0)\big)(u_n-u_0)\,dx\right| \\
&\leq\varepsilon\big(\|u_n\|^{p-1}+\|u_0\|^{p-1}\big)\|u_n-u_0\|+C_\varepsilon\big(\|u_n\|_q^{q-1}+\|u_0\|_q^{q-1}\big)\|u_n-u_0\|_q \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{2.22}$$

Define the linear function $g : X \to \mathbb{R}$ by

$$g(\omega) = \int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\nabla u_0 \nabla \omega + |u_0|^{p-2}u_0\omega \, dx. \tag{2.23}$$

Noticing that $|g(\omega)| \leq 2\|u_0\|^{p-1}\|\omega\|$, we can deduce that $g$ is continuous on $X$. Using $u_n \rightharpoonup u_0$ in $X$, we have

$$\begin{aligned} g(u_n - u_0) &= \int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\nabla u_0 \nabla (u_n - u_0) + |u_0|^{p-2}u_0(u_n - u_0)\, dx \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.24}$$

Since $\|u_n\| < \rho$, we deduce that $|Q_n| \to 0$ as $n \to \infty$.

Combining the above results, we have $|P_n| \to 0$ as $n \to \infty$, Then, using the standard inequalities in $\mathbb{R}^N$

$$\begin{aligned} &\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq C_p|x - y|^p, \quad p \geq 2, \\ &\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq \frac{C_p|x - y|^p}{|x| + |y|^{2-p}}, \quad 2 > p > 1, \end{aligned} \tag{2.25}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^N$, we can show that $u_n \to u_0$ in $X$. Thus, $u_0$ is a nontrivial weak solution of problem (1.1). The proof is completed. □

Next, we prove that problem (1.1) has a mountain-pass-type solution. To overcome the difficulty of finding a bounded $(PS)$ sequence for the associated functional $I$, motivated by [1, 10], we use a cut-off function $\psi \in C_0^1(\mathbb{R}^+)$ that satisfies

$$\begin{aligned} &\psi(t) = 1, \quad \forall t \in [0,1]; \qquad 0 \leq \psi \leq 1, \quad \forall t \in (1,2); \\ &\psi(t) \equiv 0, \quad \forall t \in [2,+\infty); \qquad \|\psi'\|_\infty \leq 2, \end{aligned} \tag{2.26}$$

and study the following modified functional $I^T$ defined by

$$I^T(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\eta_T(u)\|u\|^{p(m+1)} - \int_{\mathbb{R}^N} \bigl(F(u) + hu\bigr)\, dx, \quad u \in X, \tag{2.27}$$

where $T > 0$ and $\eta_T(u) = \psi(\frac{\|u\|^p}{T^p})$. For $T > 0$ sufficiently large and $\lambda$ sufficiently small, we will prove that there exists a critical point $\tilde{u}_0$ of $I_T$ such that $\|\tilde{u}_0\| \leq T$, and so $\tilde{u}_0$ is also a critical point of $I$. For this purpose, we use the following theorem given in [1].

**Lemma 2.3** (see[1]) *Let $X$ be a Banach space with norm $\|\cdot\|_X$, and $K \subset \mathbb{R}^+$ be an interval. Consider the family of $C^1$ functionals on $X$*

$$I_\mu(u) = A(u) - \mu B(u), \quad \mu \in K, \tag{2.28}$$

*with $B$ nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\|_X \to \infty$ and $I_\mu(0) = 0$. For any $\mu \in K$, we set*

$$\Gamma_\mu = \bigl\{\gamma \in \bigl(C[0,1], X\bigr) : \gamma(0) = 0, I_\mu\bigl(\gamma(1)\bigr) < 0\bigr\}. \tag{2.29}$$

*If for any $\mu \in K$, the set $\Gamma_\mu$ is nonempty, and*

$$c_\mu = \inf_{\gamma \in \Gamma_\mu} \max_{t \in [0,1]} I_\mu\big(\gamma(t)\big) > 0, \tag{2.30}$$

*then, for almost every $\mu \in K$, there is a sequence $\{u_n\} \subset X$ such that* (i) *$\{u_n\}$ is bounded*; (ii) $I_\mu(u_n) \to c_\mu$; (iii) $I'_\mu(u_n) \to 0$ *in* $X^{-1}$.

In our case,

$$A(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\eta_T(u)\|u\|^{p(m+1)}, \qquad B(u) = \int_{\mathbb{R}^N} \big(F(u) + hu\big)\,dx. \tag{2.31}$$

So the perturbed functional we study is

$$I_\mu^T(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\eta_T(u)\|u\|^{p(m+1)} - \mu \int_{\mathbb{R}^N} \big(F(u) + hu\big)\,dx, \tag{2.32}$$

and

$$\big(I_\mu^T(u)\big)'v = \widehat{M}\big(\|u\|\big) \int_{\mathbb{R}^N} \big(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\big)\,dx - \mu \int_{\mathbb{R}^N} \big(f(u) + h\big)v\,dx, \tag{2.33}$$

where $\widehat{M}(\|u\|) = (a + \lambda\eta_T(u)\|u\|^{pm} + \frac{\lambda}{(m+1)T^p}\eta'_T(u)\|u\|^{p(m+1)})$. The following lemmas, Lemma 2.4 and Lemma 2.5, imply that $I_\mu^T$ satisfies the conditions of Lemma 2.3.

**Lemma 2.4** *Let $(F_1)$-$(F_3)$ hold, Then $\Gamma_\mu \neq \emptyset$ for all $\mu \in [\frac{1}{2}, 1]$.*

*Proof* Choose $\beta(x) \in C_0^1(\mathbb{R}^N)$ with $\beta(x) \geq 0$ in $\mathbb{R}^N$, $\|\beta\| = 1$, and $\operatorname{supp}(\beta) \subset B_R$ for some $R > 0$. By $(F_3)$ we have that, for any $C_3 > 0$ with $C_3/2 \int_{B_R} \beta^p\,dx > a/p$, there exists $C_4 > 0$ such that

$$F(t) \geq C_3|t|^p - C_4, \quad t \in \mathbb{R}^+. \tag{2.34}$$

Then, for $t^p > 2T^p$,

$$\begin{aligned} I_\mu^T(t\beta) &= \frac{a}{p}\|t\beta\|^p + \frac{\lambda}{p(m+1)}\psi\left(\frac{\|t\beta\|^p}{T^p}\right)\|t\beta\|^{p(m+1)} - \mu \int_{\mathbb{R}^N} \big(F(t\beta) + ht\beta\big)\,dx \\ &= \frac{a}{p}\|t\beta\|^p - \mu \int_{\mathbb{R}^N} \big(F(t\beta) + ht\beta\big)\,dx \leq \left(\frac{a}{p} - \frac{C_3}{2}\int_{B_R} \beta^p\,dx\right)t^p + C_5. \end{aligned} \tag{2.35}$$

It follows that we can choose $t > 0$ large enough such that $I_\mu^T(t\beta) < 0$. The proof is completed. □

**Lemma 2.5** *Let $(F_1)$-$(F_3)$ hold. Then there exists a constant $c > 0$ such that $c_\mu \geq c > 0$ for all $\mu \in [\frac{1}{2}, 1]$ if $\|h\|_{p'} < m_1$.*

*Proof* Similarly as in the proof of Lemma 2.1, we can show that, for every $\mu \in [\frac{1}{2}, 1]$, there exists $c > 0$ such that $I_\mu^T(u) \geq c$ with $\|u\| = \tilde{\rho}$ and $\|h\|_{p'} < m_1$. Fix $\mu \in [\frac{1}{2}, 1]$ and $\gamma \in \Gamma_\mu$. By the definition of $\Gamma_\mu$, $\|\gamma(1)\| > \tilde{\rho}$. By the continuity we deduce that there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\|_E = \tilde{\rho}$. Therefore, for any $\mu \in [\frac{1}{2},1]$,

$$c_\mu = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} I_\mu^T\bigl(\gamma(t)\bigr) \geq \inf_{\gamma\in\Gamma_\mu} I_\mu^T\bigl(\gamma(t_\gamma)\bigr) \geq c > 0, \tag{2.36}$$

which completes the proof. □

**Lemma 2.6** *For any $\mu \in [\frac{1}{2},1]$ and $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$, each bounded (PS) sequence of the functional $I_\mu^T$ admits a convergent subsequence.*

*Proof* By Lemmas 2.3-2.5, we obtain that, for a.e. $\mu \in [1/2,1]$, there is a bounded sequence $\{u_n\}$ in $X$ that satisfies

$$I_\mu^T(u_n) \to c_\mu, \qquad \bigl(I_\mu^T(u_n)\bigr)' \to 0 \quad \text{in } X^*, \quad \text{and} \quad \sup_n \|u_n\| < T. \tag{2.37}$$

Since the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $q \in (p,p^*)$, passing to a subsequence, we can assume that

$$u_n \rightharpoonup u, \quad \text{in } X; \qquad u_n \to u, \quad L^q\bigl(\mathbb{R}^N\bigr); \qquad u_n \to u, \quad \text{a.e. in } \mathbb{R}^N. \tag{2.38}$$

By (2.16) we can get

$$\bigl(I_\mu^T(u_n) - I_\mu^T(u)\bigr)'(u_n - u) = A_n + B_n + \mu C_n, \tag{2.39}$$

where

$$\begin{aligned} A_n &= \widehat{M}(u_n) \int_{\mathbb{R}^N} \bigl(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\bigr)\nabla(u_n - u) \\ &\quad + \bigl(|u_n|^{p-2}u_n - u^{p-2}u\bigr)(u_n - u)\,dx, \\ B_n &= \bigl(\widehat{M}(u_n) - \widehat{M}(u)\bigr) \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla(u_n - u) + |u|^{p-2}u(u_n - u)\,dx, \\ C_n &= \int_{\mathbb{R}^N} \bigl(f(u_n) - f(u)\bigr)(u_n - u)\,dx. \end{aligned} \tag{2.40}$$

It is clear that

$$\bigl(I_\mu^T(u_n) - I_\mu^T(u)\bigr)'(u_n - u) \to 0 \quad \text{as } n \to \infty. \tag{2.41}$$

An analogous argument as in (2.22) and (2.25) gives us that

$$B_n \to 0 \quad \text{and} \quad C_n \to 0 \quad \text{as } n \to \infty. \tag{2.42}$$

Combining the above results and $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$, we have that $|A_n| \to 0$ as $n \to \infty$. Then, using a standard equality ([3], Lemma 2.1), we can show that $u_n \to u$ in $X$. The proof is completed. □

**Lemma 2.7** *Assume* $(F_1)$-$(F_3)$ *and* $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$. *Then, for almost every* $\mu \in [\frac{1}{2}, 1]$, *there exist* $u^\mu \in X \setminus \{0\}$ *such that* $(I_\mu^T)'(u^\mu) = 0$ *and* $I_\mu^T(u^\mu) = c_\mu$ *with* $\|h\|_{p'} < m_1$.

*Proof* It follows from Lemmas 2.3-2.5 that, for every $\mu \in [\frac{1}{2}, 1]$, there exists a bounded sequence $\{u_n^\mu\} \subset X$ such that

$$I_\mu^T(u_n^\mu) \to c_\mu \quad \text{and} \quad (I_\mu^T)'(u_n^\mu) \to 0 \quad \text{as } n \to \infty.$$

By Lemma 2.6 we can suppose that $u^\mu \in X$ and $u_n^\mu \to u^\mu$ in $X$. The proof is completed. □

According to Lemma 2.6, there exists a sequence $\{\mu_n\} \subset [\frac{1}{2}, 1]$ with $\mu_n \to 1$ and $\{u_n\} \subset X$ as $n \to \infty$ such that $I_{\mu_n}^T(u_n) = c_{\mu_n}$, $(I_{\mu_n}^T)'(u_n) = 0$, and $u_n$ is a positive solution of

$$\widehat{M}(\|u\|)(-\Delta_p u + |u|^{p-2}u) = \mu_n(f(u) + h(x)). \tag{2.43}$$

In the following, to obtain $\|u_n\| < T$, we establish an identity that extends the Kazin-Pohozav identity in ([13], Thm. 29.4) with $p = 2$.

**Lemma 2.8** *Assume that* $f(x,u) : \mathbb{R}^N \times \mathbb{R}^1 \to \mathbb{R}^1$ *is a Caretheódary function,* $u \in C^2_{\text{loc}}(\mathbb{R}^N)$ *is a solution of*

$$\begin{cases} -\Delta_p u + f(x,u) = 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } \to 0, \end{cases} \tag{2.44}$$

$\frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N)$, $i = 1, 2, \ldots$, *and* $F(x,u), F_1(x,u) \in L^1(\mathbb{R}^N)$. *Then*

$$\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} (NF(x,u) + F_1(x,u)) \, dx = 0, \tag{2.45}$$

*where* $F(x,u) = \int_0^u f(x,s) \, ds$ *and* $F_1(x,u) = \sum_{i=1}^N x_i \frac{\partial F(x,u)}{\partial x_i}$.

*Proof* Multiplying equation (2.44) by $x \cdot \nabla u$ and integrating over the ball $B_R$, we obtain

$$\int_{B_R} f(x,u) x \cdot \nabla u \, dx = \int_{B_R} \operatorname{div}(|\nabla u|^{p-2} \nabla u) x \cdot \nabla u \, dx. \tag{2.46}$$

Then

$$\begin{aligned} \int_{B_R} f(x,u) x \cdot \nabla u \, dx &= \sum_{i=1}^N \int_{B_R} x_i f(x,u) \frac{\partial u}{\partial x_i} \, dx \\ &= \sum_{i=1}^N \int_{B_R} \left( \frac{\partial}{\partial x_i}(x_i F(x,u)) - \left( F(x,u) + x_i \frac{\partial F(x,u)}{\partial x_i} \right) \right) dx \\ &= \sum_{i=1}^N \int_{\partial B_R} F(x,u) x_i n_i \, ds - \int_{B_R} (NF(x,u) + F_1(x,u)) \, dx \\ &= R \int_{\partial B_R} F(x,u) \, ds - \int_{B_R} (NF(x,u) + F_1(x,u)) \, dx, \end{aligned} \tag{2.47}$$

where $n_i$ are the components of the unit outward normal to $\partial B_R$, and $ds$ is an area element. On the other hand, integrating by parts, we obtain

$$\begin{aligned}
&\int_{B_R} \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)x\cdot\nabla u\,dx \\
&= \sum_{j=1}^{N}\int_{B_R}\frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\right)\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\,dx \\
&= \sum_{j=1}^{N}\int_{B_R}\left(\frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\right)-|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\left(\frac{\partial}{\partial x_j}\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\right)\right)dx \\
&= \int_{\partial B_R}|\nabla u|^{p-2}\frac{\partial u}{\partial n}x\cdot\nabla u\,ds-\int_{B_R}|\nabla u|^{p}\,dx \\
&\quad -\int_{B_R}\sum_{j=1}^{N}|\nabla u|^{p-2}\left(\sum_{i=1}^{N}x_i\frac{\partial^2 u}{\partial x_i\,\partial x_j}\frac{\partial u}{\partial x_j}\right)dx.
\end{aligned} \tag{2.48}$$

On $B_R$, we have $\nabla u = \frac{\partial u}{n}\cdot\vec{n} = \frac{\partial u}{\partial n}\frac{x}{R}$ and

$$\int_{\partial B_R}|\nabla u|^{p-2}\frac{\partial u}{\partial n}x\cdot\nabla u\,dx = R\int_{\partial B_R}|\nabla u|^{p}\,ds. \tag{2.49}$$

Further, we have

$$\begin{aligned}
&\int_{B_R}\sum_{j=1}^{N}|\nabla u|^{p-2}\left(\sum_{i=1}^{N}x_i\frac{\partial^2 u}{\partial x_i\,\partial x_j}\frac{\partial u}{\partial x_j}\right)dx \\
&= \frac{1}{p}\sum_{i=1}^{N}\int_{B_R}\left(\frac{\partial}{\partial x_i}\big(x_i|\nabla u|^{p}\big)-|\nabla u|^{p}\right)dx \\
&= \frac{R}{p}\int_{\partial B_R}|\nabla u|^{p}\,ds-\frac{N}{p}\int_{B_R}|\nabla u|^{p}\,dx.
\end{aligned} \tag{2.50}$$

Therefore, we obtain

$$R\int_{\partial B_R}\left(F-\left(1-\frac{1}{p}\right)|\nabla u|^{p}\right)ds+\left(1-\frac{N}{p}\right)\int_{B_R}|\nabla u|^{p}\,dx-\int_{B_R}(NF+F_1)\,dx=0. \tag{2.51}$$

Since $F(x,u)\in L^1(\mathbb{R}^N)$ and $u\in X$, we claim that

$$\liminf_{n\to\infty} R\int_{\partial B_R}\big(\big|F(x,u)\big|+|\nabla u|^{p}\big)\,dS=0. \tag{2.52}$$

Indeed, otherwise,

$$\liminf_{n\to\infty} R\int_{\partial B_R}\big(\big|F(x,u)\big|+|\nabla u|^{p}\big)\,dS=a_0>0. \tag{2.53}$$

Then, there exists $R_0>0$ such that, for $R\geq R_0$,

$$R\int_{\partial B_R}\big(\big|F(x,u)\big|+|\nabla u|^{p}\big)\,dS\geq\frac{a_0}{2}. \tag{2.54}$$

Let $R_n = R_0 + n$, $n = 1, 2, \ldots$. Then $R_n \to \infty$ as $n \to \infty$. It follows from the integral mean theorem that there is $\xi_n \in (R_{n-1}, R_n)$ and $\xi_n \geq R_0$ such that, for $R \geq R_0$,

$$\int_{R_{n-1}}^{R_n} \int_{\partial B_R} \left(|F| + |\nabla u|^p\right) ds\, dR = \xi_n \int_{\partial B_{\xi_n}} \left(|F| + |\nabla u|^p\right) ds \geq \frac{a_0}{2}, \tag{2.55}$$

and thus

$$\int_{R_0}^{\infty} \int_{\partial B_R} \left(|F| + |\nabla u|^p\right) ds\, dR \geq \sum_{n=2}^{\infty} \int_{R_{n-1}}^{R_n} \int_{\partial B_R} \left(|F| + |\nabla u|^p\right) ds\, dR = \infty. \tag{2.56}$$

This contradicts the fact

$$\int_{\mathbb{R}^N} \left(|F| + |\nabla u|^p\right) dx = \int_0^{\infty} \int_{\partial B_R} \left(|F| + |\nabla u|^p\right) ds\, dR < \infty. \tag{2.57}$$

Therefore, (2.52) is true. Thus, letting $R \to \infty$ in (2.51), we have

$$\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla u|^p\, dx + \int_{\mathbb{R}^N} \left(NF(x,u) + F_1(x,u)\right) dx = 0. \tag{2.58}$$

Then, we finish the proof of Lemma 2.8. □

**Lemma 2.9** *Let $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$, and let $u \in X$ be a weak solution of*

$$\widehat{M}\left(\|u\|\right)\left(-\Delta_p u + |u|^{p-2}u\right) = \mu\left(f(u) + h(x)\right), \tag{2.59}$$

*where $\widehat{M}(\|u\|) = (a + \lambda \eta_T(u)\|u\|^{pm} + \frac{\lambda}{(m+1)T^p}\eta_T'(u)\|u\|^{p(m+1)})$. Then the following identity holds*:

$$\begin{aligned} &\widehat{M}\left(\|u\|\right)\left(\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \frac{N}{p}\int_{\mathbb{R}^N} |u|^p\, dx\right) \\ &\quad = N\mu \int_{\mathbb{R}^N} \left(F(u) + hu\right) dx + \mu \int_{\mathbb{R}^N} \nabla h \cdot xu\, dx. \end{aligned} \tag{2.60}$$

*Proof* Since $u \in X$ is a weak solution of (2.59), by standard regularity results, $u \in C^2_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$. Let

$$g(x,u) = \frac{\mu(f(u) + h(x))}{\widehat{M}(\|u\|)} - |u|^{p-2}u. \tag{2.61}$$

Then $u \in X$ is also a solution of

$$-\Delta_p u = g(x,u). \tag{2.62}$$

By Lemma 2.8,

$$\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla u|^p\, dx = \int_{\mathbb{R}^N} \left(NG(u) + G_1(x,u)\right) dx, \tag{2.63}$$

where $G(x,u) = \int_0^u g(x,s)\, ds$ and $G_1(x,u) = \sum_{i=1}^N x_i \frac{\partial G(x,u)}{\partial x_i}$. Then the conclusion holds. □

**Lemma 2.10** *Assume that* $(F_1)$-$(F_3)$ *and* $(H)$ *hold and that* $\|h\|_{p'} < m_1$ *for* $m_1$ *given in Lemma* 2.6. *Let* $u_n$ *be a critical point of* $I_{\mu_n}^T$ *at level* $c_{\mu_n}$. *Then for T sufficiently large, there exists* $\lambda_0 = \lambda_0(T)$ *with* $\lambda_0 < a(\frac{m+1}{m+3})T^{-pm}$ *such that, for any* $\lambda \in [0, \lambda_0)$, *subject to a subsequence,* $\|u_n\| < T$ *for all* $n \in \mathbb{N}$.

*Proof* Since $(I_{\mu_n}^T)'(u_n) = 0$, by Lemma 2.9 $u_n$ satisfies

$$\begin{aligned} &\widehat{M}\big(\|u\|\big)\left(\frac{N}{p}\|u\|^p + \int_{\mathbb{R}^N} |\nabla u|^p\,dx\right) \\ &\quad = N\mu_n \int_{\mathbb{R}^N} \big(F(u_n) + hu_n\big)\,dx + \mu_n \int_{\mathbb{R}^N} \nabla h \cdot xu_n\,dx. \end{aligned} \tag{2.64}$$

Using $I_{\mu_n}^T(u_n) = c_{\mu_n}$, we have

$$\frac{aN}{p}\|u_n\|^p + \frac{\lambda N}{p(m+1)}\eta_T(u_n)\|u_n\|^{p(m+1)} = N\mu_n \int_{\mathbb{R}^N} \big(F(u_n) + hu_n\big)\,dx + Nc_{\mu_n}. \tag{2.65}$$

Therefore, by (2.64), (2.65) and $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$ we deduce that

$$\begin{aligned} &\frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx \\ &\quad \le \widehat{M}\big(\|u_n\|\big)\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx \\ &\quad = Nc_{\mu_n} + N\left(\widehat{M}\big(\|u_n\|\big) - \frac{a}{p}\right)\|u\|^p - \frac{\lambda N}{p(m+1)}\eta_T(u_n)\|u_n\|^{p(m+1)} - \mu_n \int_{\mathbb{R}^N} \nabla hx \cdot u_n\,dx \\ &\quad \le Nc_{\mu_n} + \frac{\lambda Nm}{p(m+1)}\eta_T(u_n)\|u_n\|^{p(m+1)} + \frac{\lambda N}{p(m+1)T^p}\eta_T'(u_n)\|u_n\|^{p(m+2)} \\ &\quad - \mu_n \int_{\mathbb{R}^N} \nabla hx \cdot u_n\,dx. \end{aligned} \tag{2.66}$$

By the min-max definition of the mountain pass level, Lemma 2.5, and (2.35) we have

$$\begin{aligned} c_{\mu_n} &\le \max_t I_{\mu_n}^T(t\beta) \\ &\le \max_t \left\{\left(\frac{a}{p} - \frac{C_3}{2}\int_{B_R} |\beta|^p\,dx\right)t^p + C_5\right\} + \max_t \frac{\lambda}{p(m+1)}\psi\left(\frac{t^p}{T^p}\right)t^{p(m+1)} \\ &\le \frac{\lambda 2^{m+1}}{p(m+1)}T^{p(m+1)} + C_5. \end{aligned} \tag{2.67}$$

Using $(H)$ and the Young equality, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla h \cdot xu_n\,dx &\le \frac{1}{p'}\int_{\mathbb{R}^N} |\xi|^{p'}\,dx + \frac{1}{p}\int_{\mathbb{R}^N} |\xi|^{p'}|u_n|^p\,dx \\ &\le \frac{1}{p}\int_{\mathbb{R}^N} |\xi|^{p'}|u_n|^p\,dx + C_6. \end{aligned} \tag{2.68}$$

We can easily calculate that

$$\eta_T(u_n)\|u_n\|^{p(m+1)} \le 2^{m+1}T^{p(m+1)}, \qquad \eta'(u_n)\|u_n\|^{p(m+2)} \le 2^{m+2}T^{p(m+2)}. \tag{2.69}$$

Combining the above estimates, we see that

$$\frac{a}{2}\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx \le \frac{\lambda N(m+5)}{p(m+1)}2^{m+1}T^{p(m+1)} + \frac{1}{p}\int_{\mathbb{R}^N}|\xi|^{p'}|u_n|^p\,dx + C_7. \tag{2.70}$$

Since $\xi(x)\in L^{p'}(\mathbb{R}^N)\cap W^{1,\infty}$, we see that $\xi^{p'}u_n\in X$. It follows from $(I^T_{\mu_n}(u_n))'(\xi^{p'}u_n)=0$ that

$$\begin{aligned}&\widehat{M}\big(\|\xi^{p'}u_n\|\big)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\big(\xi^{p'}u_n\big)+|u_n|^{p-2}u\big(\xi^{p'}u_n\big)\,dx\\&\quad=\mu_n\int_{\mathbb{R}^N}\big(f(u_n)+h\big)\xi^{p'}u_n\,dx.\end{aligned} \tag{2.71}$$

Since $a>2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$, we have $(3a/2)\ge\widehat{M}(\|\xi^{p'}u_n\|)$, and it follows from (2.69) and (2.71) that

$$(3a/2)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\big(\xi^{p'}u_n\big)+|u_n|^p\xi^{p'}\,dx\ge(1/2)\int_{\mathbb{R}^N}f(u_n)u_n\xi^{p'}\,dx. \tag{2.72}$$

From (2.70) by the Hölder inequality we deduce that

$$\begin{aligned}&3a\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\big(\xi^{p'}u_n\big)\,dx\\&\quad\le 3a\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\big(p'\xi^{p'-1}u_n\nabla\xi+\xi^{p'}\nabla u_n\big)\,dx\\&\quad\le 3\big(\|\xi\|^{\infty}_{p'}+\|\nabla\xi\|^{\infty}_{p'}\big)\left(a\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx\right)+3a(p-1)^{-1}\int_{\mathbb{R}^N}\xi^{p'}|u_n|^p\,dx\\&\quad\le C\lambda T^{p(m+1)}+C\int_{\mathbb{R}^N}\xi^{p'}|u_n|^p\,dx+C,\end{aligned} \tag{2.73}$$

where $C$ is a constant independent of $\lambda$ and $T$.

By $(F_3)$, for any $L>0$, there exists $C(L)>0$ such that

$$f(s)s\ge Ls^p-C(L)\quad\text{for all } s>0. \tag{2.74}$$

Combining (2.72)-(2.74), we get

$$\left(\frac{1}{2}L-C\right)\int_{\mathbb{R}^N}\xi^{p'}|u_n|^p\,dx\le C\lambda T^{p(m+1)}+C. \tag{2.75}$$

For $L>0$ large enough, we obtain

$$\int_{\mathbb{R}^N}\xi^{p'}|u_n|^p\,dx\le C\lambda T^{p(m+1)}+C. \tag{2.76}$$

It follows from (2.70) and (2.76) that

$$\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx\le C\lambda T^{p(m+1)}+C. \tag{2.77}$$

On the other hand,

$$\begin{aligned} &a\|u_n\|^p + \eta_T(u_n)\|u_n\|^{p(m+1)} + \frac{\lambda}{m+1}\eta'_T(u_n)\|u_n\|^{p(m+2)} \\ &\quad = \mu_n \int_{\mathbb{R}^N} \big(f(u_n)u_n + hu_n\big)\,dx \\ &\quad \le \varepsilon\|u_n\|^p + C_\varepsilon\|u_n\|_{p^*}^{p^*} + \frac{1}{p'}\|h\|_{p'}^{p'} + \frac{1}{p}\|u\|^p. \end{aligned} \tag{2.78}$$

By (2.77) and (2.78) we have

$$\begin{aligned} (a-\varepsilon-1/p)\|u_n\|^p &\le C_\varepsilon\|u_n\|_{p*}^{p*} - \lambda/\big((m+1)T^p\big)\eta'_T(u_n)\|u_n\|^{p(m+2)} + C \\ &\le C\|\nabla u_n\|_p^{p*} + \lambda 2^{m+2}(m+1)^{-1}T^{p(m+1)} + C \\ &\le C\lambda T^{p^*(m+1)} + C\lambda T^{p(m+1)} + C. \end{aligned} \tag{2.79}$$

Suppose that $\|u_n\| > T$ for $n \in \mathbb{N}$ and $T$ large enough. Then

$$T^p < \|u_n\|^p \le C\lambda T^{p^*(m+1)} + C\lambda T^{p(m+1)} + C, \tag{2.80}$$

which is not true if we choose $T$ large and $\lambda$ small enough. So by setting $\lambda(T)$ small we obtain that the sequence $\{u_n\}$ is bounded for any $\lambda \in [0, \lambda_0)$, and the conclusion holds. □

**Lemma 2.11** *Let $T$, $\lambda_0$ be defined by Lemma* 2.10, *and $u_n$ be the critical point of $I_{\mu_n}^T$ at level $c_{\mu_n}$. Then the sequence $\{u_n\}$ is also a (PS) sequence for $I$.*

*Proof* From the proof of Lemma 2.10 we may assume that $\|u_n\| \le T$. So

$$I(u_n) = I_{\mu_n}^T(u_n) + (\mu_n - 1)\int_{\mathbb{R}^N} \big(F(u_n) + hu_n\big)\,dx. \tag{2.81}$$

Since $\mu_n \to 1$, we can show that $\{u_n\}$ is a $(PS)$ sequence of $I$. Indeed, the boundedness of $\{u_n\}$ implies that $\{I_{\mu_n}^T\}$ is bounded. Also,

$$I'(u_n)v = \big(I_{\mu_n}^T\big)'(u_n, v) + (\mu_n - 1)\int_{\mathbb{R}^N} \big(f(u_n) + h(u_n)\big)v\,dx, \quad v \in X. \tag{2.82}$$

Thus, $I'(u_n) \to 0$, and $\{u_n\}$ is a bounded $(PS)$ sequence of $I$. By Lemma 2.5, $\{u_n\}$ has a convergent subsequence. We may assume that $u_n \to \tilde{u}_0$. Consequently, $I'(\tilde{u}_0) = 0$. According to Lemma 2.4, we have that $I(\tilde{u}_0) = \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} I_{\mu_n}^T(u_n) \ge c > 0$ and $\tilde{u}_0$ is a solution of problem (1.1). Thus, we completed the proof. □

*Proof of Theorem* 1.1 By Lemma 2.2 the problem has a solution $u_0 \in X$ with $I(u_0) < 0$. From Lemma 2.9 we know that problem (1.1) possesses a second solution $\tilde{u}_0 \in X$ with $I(\tilde{u}_0) \ge c > 0$. Hence, $u_0 \ne \tilde{u}_0$, and we complete the proof of Theorem 1.1. □

## 3 Proof of Theorem 1.2

Let $I_\lambda(u): X \to \mathbb{R}$ be the energy functional associated with problem (1.13) defined by

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\|u\|^{p(m+1)} - \frac{1}{q}\|u\|_q^q, \tag{3.1}$$

where $F(u) = \int_0^u f(s)\,ds$. It is easy to see that the functional $I \in C^1(E,\mathbb{R})$ and its Gateaux derivative is given by

$$I'_\lambda(u)v = \big(a + \lambda\|u\|^{pm}\big)\int_{\mathbb{R}^N}\big(|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv\big)\,dx \\ - \int_{\mathbb{R}^N}|u|^{q-2}uv\,dx, \quad \forall v \in E. \tag{3.2}$$

Clearly, we see that a weak solution of (1.13) corresponds to a critical point of the functional.

In this part, we first proof the nonexistence for problem (1.13) for large $\lambda > \lambda^*$. which means that if a solution exists, then $\lambda$ must sufficiently small. Secondly, we obtain that there exists $\lambda^{**}$ such that problem (1.1) has at least one solution for any $0 < \lambda < \lambda^{**}$. Finally, by the properties of $\lambda^*$ and $\lambda^{**}$ we deduce that $\lambda^* = \lambda^{**}$. We will break the proof into six steps.

*Proof of Theorem* 1.2 *Step 1. Nonexistence for large* $\lambda > 0$. It is sufficient to show that if $u$ is a nontrivial solution of problem (1.13), then $\lambda > 0$ must be small. Assume that $u$ is a nontrivial solution of problem (1.1). Then we get $I'_\lambda(u)u = 0$, that is,

$$a\|u\|^p + \lambda\|u\|^{p(m+1)} = \|u\|_q^q. \tag{3.3}$$

Since $p < q < \min\{p(m+1), p^*\}$, applying the Young inequality and (1.9), we deduce that

$$a\|u\|^p + \lambda\|u\|^{p(m+1)} = \|u\|_q^q \le S_q^q\|u\|_E^q \le a\|u\|_E^p + \lambda_1\|u\|_E^{p(m+1)}, \tag{3.4}$$

which implies that $\lambda \le \lambda_1 = (S_q^q)^{\frac{pm}{q-p}} a^{-\frac{p(m+1)-q}{q-p}}$. On the other hand, if $\lambda^* \ge \lambda_1$, then we conclude that problem (1.1) has no solution for any $\lambda \in (\lambda^*, +\infty)$.

*Step 2. Coercivity of* $I_\lambda(u)$. Indeed, for any $u \in E$ and all $\lambda > 0$,

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\|u\|^{p(m+1)} - \frac{1}{q}\|u\|_q^q \\ \ge \frac{a}{p}\|u\|^p + \frac{\lambda}{2p(m+1)}\|u\|^{p(m+1)} + \frac{\lambda}{2p(m+1)}\|u\|^{p(m+1)} - \frac{S_q^q}{q}\|u\|^q. \tag{3.5}$$

Since $q < p(m+1)$, there exists $C_1 = C_1(\lambda, q, m, S_q)$ such that

$$\frac{S_q^q}{q}\|u\|^q \le \frac{\lambda}{2p(m+1)}\|u\|^{p(m+1)} + C_1. \tag{3.6}$$

It follows that

$$I_\lambda(u) \ge \frac{a}{p}\|u\|^p + \frac{\lambda}{2p(m+1)}\|u\|^{p(m+1)} - C_1. \tag{3.7}$$

This implies that $I_\lambda(u)$ is coercive.

*Step 3. The infimum of* $I_\lambda$ *is attained.* Let $\{u_n\}$ be a minimizing sequence of $I_\lambda$. Then from Step 2 we immediately see that $\{u_n\}$ is bounded in $X$ . Therefore, without loss of generality, we may assume that $\{u_n\}$ is nonnegative and converges weakly and pointwise to some $u$ in $X$.

Using the compact embedding $X \hookrightarrow L^q(\mathbb{R}^N)$, we have

$$\|u\|_q = \lim_{n\to\infty} \|u_n\|_q \quad \text{and} \quad \|u\| \le \liminf_{n\to\infty} \|u_n\| \tag{3.8}$$

by the weak lower semicontinuity of the norm $\|\cdot\|$. Thus,

$$\begin{aligned} I_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\|u\|^{p(m+1)} - \frac{1}{q}\|u\|_q^q \\ &\le \liminf_{n\to\infty}\left(\frac{a}{p}\|u_n\|^p + \frac{\lambda}{p(m+1)}\|u_n\|^{p(m+1)}\right) - \frac{1}{q}\lim_{n\to\infty}\|u_n\|_q^q \\ &\le \liminf_{n\to\infty}\left(\frac{a}{p}\|u_n\|^p + \frac{\lambda}{p(m+1)}\|u_n\|^{p(m+1)} - \frac{1}{q}\|u_n\|_q^q\right) = \liminf_{n\to\infty} I_\lambda(u_n). \end{aligned} \tag{3.9}$$

Therefore, $u$ is a global minimum for $I_\lambda$, and hence it is a critical point, namely a weak solution to problem (1.1).

*Step 4. The weak solution u is nontrivial if* $\lambda > 0$ *is sufficiently small.* Clearly, $I_\lambda(0) = 0$. Therefore, it is sufficient to show that there exists $\lambda_0 > 0$ such that

$$\inf_{u\in E} I_\lambda(u) < 0, \quad \text{for any } \lambda \in (0, \lambda_0). \tag{3.10}$$

Choose $u_0 \in C_0^\infty(\mathbb{R}^N)$, $u_0 \not\equiv 0$, such that $\|u_0\|_E = 1$. Denote

$$I_\lambda(tu_0) = t^p s(t), \qquad s(t) = B_1 + \lambda B_2 t^{pm} - B_3 t^{q-p}, \quad t \ge 0, \tag{3.11}$$

where

$$B_1 = \frac{a}{p}, \qquad B_2 = \frac{1}{p(m+1)} > 0, \qquad B_3 = \frac{1}{q}\int_{\mathbb{R}^N} |u_0|^q\,dx > 0.$$

Then there exist $\lambda_0 > 0$ and large $t_\lambda > 0$ such that $I_\lambda(t_\lambda u_0) < 0$ for $\lambda \in (0, \lambda_0]$. Let $e = t_\lambda u_0$. Then $\|e\| = t_\lambda$ and $I_\lambda(e) < 0$. This implies that (3.10) is true. So the weak solution $u$ is nontrivial if $\lambda > 0$ is sufficiently small.

Now, we define

$$\begin{aligned} &\lambda^{**} = \sup\{\lambda > 0, \text{problem (1.13) admits a nontrival weak solution}\}, \\ &\lambda^{*} = \inf\{\lambda > 0, \text{problem (1.13) does not admit any nontrival weak solution}\}. \end{aligned}$$

Clearly, $\lambda^{**} \ge \lambda^*$. To complete the proof of Theorem 1.2, it suffices to prove the following facts: (a) problem (1.13) has a weak solution for any $\lambda < \lambda^{**}$; (b) $\lambda^{**} = \lambda^*$, and problem (1.13) admits a weak solution when $\lambda = \lambda^*$.

*Step 5. Problem (*1.13*) has a solution for any* $\lambda < \lambda^{**}$ *and* $\lambda^* = \lambda^{**}$. Fix $\lambda < \lambda^{**}$. By the definition of $\lambda^{**}$, there exists $\mu \in (\lambda, \lambda^{**})$ such that $I_\lambda$ has a nontrivial critical point $u_\mu \in E$. Clearly, we have

$$\left(a + \lambda\left(\int_{\mathbb{R}^N}\left(|\nabla u_m u|^p + |u_\mu|^p\right)dx\right)^m\right)\left(-\Delta_p u_\mu + |u_\mu|^{p-2}u_\mu\right) \le |u_\mu|^{q-2}u_\mu. \tag{3.12}$$

This implies that $u_\mu$ is a subsolution of problem (1.13). In order to find a supsolution of (1.13) that dominates $u_\mu$, we consider the constrained minimization problem

$$\inf\left\{\frac{a}{p}\|\omega\|^p + \frac{\lambda}{p(m+1)}\|\omega\|^{p(m+1)} - \frac{1}{q}\|\omega\|_q^q : \omega \in E, \|\omega\|_q^q = q \text{ and } \omega \geq u_\mu\right\}. \tag{3.13}$$

Arguments similar to those used in Step 3 and Step 4 show that the above minimization has a solution $u_\lambda \geq u_\mu$, which is also a weak solution of problem (1.13). Hence, problem (1.13) admits a weak solution for any $\lambda \in [0, \lambda^{**})$, This means that $\lambda^* \geq \lambda^{**}$ by the definition of $\lambda^*$. But we already know that $\lambda^{**} \geq \lambda^*$, and therefore $\lambda^{**} = \lambda^*$.

*Step 6. Problem (*1.13*) admits a nontrivial solution when* $\lambda = \lambda^*$. Let $\{\lambda_n\}$ be a increasing sequence converging to $\lambda^*$, and $\{u_n\}$ be a sequence of solutions of (1.1) corresponding to $\lambda_n$. By Step 2, $\{u_n\}$ is bounded in $X$, and without loss of generality we may assume that $u_n \rightharpoonup u$ in $X$, $u_n \to u$ in $L^q(\mathbb{R}^N)$, and $u_n \to u^*$ a.e. in $X$. It follows from $I_\lambda(u_n)v = 0$ that, for any $v \in X$,

$$\left(a + \lambda_n\|u_n\|^{pm}\right)\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n\nabla v + |u_n|^{p-2}u_n v\right)dx = \int_{\mathbb{R}^N}|u_n|^{q-2}u_n v\,dx. \tag{3.14}$$

Then, passing to the limit as $n \to \infty$, we deduce that $u^*$ satisfies $I_\lambda(u^*)v = 0$ when $\lambda = \lambda^*$. Now, it remains to prove that $u^*$ is a nontrivial critical point for $I_{\lambda^*}$. From $I'_\lambda(u_n)u_n = 0$ it is easy to deduce that $\|u_n\| \geq (\lambda_n S_q^{-q})^{1/(q-p(m+1))}$, which implies that $u_n$ has a lower bound. Next, since $\lambda_n \nearrow \lambda^*$ as $n \to \infty$, it suffices to show that $\|u_n - u^*\| \to 0$ as $n \to \infty$.

Since $u_n$ and $u^*$ are the solutions of (1.1) corresponding to $\lambda_n$ and $\lambda^*$, we see that

$$0 = \left(I'_{\lambda_n}(u_n) - I'_{\lambda^*}\left(u^*\right)\right)(u_n - u) = X_n + Y_n - Z_n, \tag{3.15}$$

where

$$\begin{aligned}
X_n &= \left(a + \lambda_n\|u_n\|^{pm}\right)\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n - \left|\nabla u^*\right|^{p-2}\nabla u^*\right)\nabla\left(u_n - u^*\right)dx \\
&\quad + \left(|u_n|^{p-2}u_n - \left|u^*\right|^{p-2}u^*\right)\left(u_n - u^*\right)dx; \\
Y_n &= \left(\lambda_n\|u_n\|^{pm} - \lambda^*\left\|u^*\right\|^{pm}\right)\int_{\mathbb{R}^N}\left|\nabla u^*\right|^{p-2}\nabla u^*\nabla\left(u_n - u^*\right) \\
&\quad + \left|u^*\right|^{p-2}u^*\left(u_n - u^*\right)dx; \\
Z_n &= \int_{\mathbb{R}^N}\left(|u_n|^{q-2}u_n - \left|u^*\right|^{q-2}u^*\right)\left(u_n - u^*\right)dx.
\end{aligned}$$

By the Hölder inequality and compact embedding $u_n \to u$ in $L^q(\mathbb{R}^N, H)$ we have

$$\begin{aligned}
|X_n| &= \left|\int_{\mathbb{R}^N}\left(|u_n|^{q-2}u_n - \left|u^*\right|^{q-2}u^*\right)\left(u_n - u^*\right)dx\right| \\
&\leq \int_{\mathbb{R}^N}\left(|u_n|^{q-1} + \left|u^*\right|^{q-1}\right)\left|u_n - u^*\right|dx \\
&\leq C\left(\|u_n\|^{q-1} + \left\|u^*\right\|^{q-1}\right)\left\|u_n - u^*\right\|_q \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.16}$$

Next, consider the functional $j : X \to \mathbb{R}$ defined by

$$j(\omega) = \int_{\mathbb{R}^N} \left|\nabla u^*\right|^{p-2} \nabla u^* \nabla \omega + \left|u^*\right|^{p-2} u^* \omega \, dx. \tag{3.17}$$

Since $|j(\omega)| \le 2\|u^*\|^{p-1}\|\omega\|$, $j$ is continuous on $X$. Using $u_n \rightharpoonup u^*$ and the boundedness of $u_n$ and $u^*$ in $X$, we have that

$$|Y_n| \le \left(\|u_n\|^{pm} + \left\|u^*\right\|^{pm}\right)\left|g\left(u_n - u^*\right)\right| \to 0 \quad \text{as } n \to \infty. \tag{3.18}$$

Combining (3.15), (3.16), and (3.18), this forces $X_n \to 0$ as $n \to \infty$. Then, using the standard inequality (2.25) in $\mathbb{R}^N$, we have that $\|u_n - u^*\| \to 0$ as $n \to \infty$, and thus $u^*$ is a nontrivial weak solution of problem (1.13) corresponding to $\lambda = \lambda^*$. This completes the proof of Theorem 1.2. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
Each of the authors contributed to each part of this study equally. Both authors read and proved the final vision of the manuscript.

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